Some more properties of the bisect-diagonal quadrilateral

MICHAEL DE VILLIERS

Martin Josefsson [1] has coined the term ‘bisect-diagonal quadrilateral’ for a quadrilateral with at least one diagonal bisected by the other diagonal, and extensively explored some of its properties. This quadrilateral has also been called a ‘bisecting quadrilateral’ [2], a ‘sloping-kite’ or ‘sliding-kite’ [3], or ‘slant kite’ [4]. The purpose of this paper is to explore some more properties of this quadrilateral.

A familiar property of the bisect-diagonal quadrilateral that is proved in Coxeter [5, pp. 54-55) as well as in Josefsson [1, pp. 215], and is extended to the concave case by Pillay & Pillay [6, pp. 16-17], is the following:

Theorem 1: A quadrilateral is a bisect-diagonal quadrilateral (where at least one diagonal bisects the other) if, and only if, the diagonal that bisects the other also bisects the area of the quadrilateral.

Equipartitioning point of a quadrilateral

As is well known, the centroid G of a triangle ABC divides, or equipartitions, the triangle into three triangles, AGB, BCG and CGA, of equal area.

The question now arises whether one can find a similar point P for a quadrilateral $ABCD$ that divides, or equi-partitions, it into four triangles, APB, BPC, CPD and DPA, of equal area. For a parallelogram, it's obvious that such an ‘*equipartitioning*’ point P exists, and would be located at its centroid, i.e. the intersection of its diagonals. But what about a more general quadrilateral? Where can P be located?

Based on the example of the triangle and the parallelogram, one may intuitively feel that in general such a point would be located at either the point mass centroid or the lamina centroid of a quadrilateral. However, a quick experimental check using an accurately constructed sketch with dynamic geometry as shown in Figure 1, shows that neither the point mass centroid* G_{PM} nor the lamina centroid† G_L respectively divide the quadrilaterals $ABCD$ and $KLMN$ into four triangles of equal area. Since G_L is the balancing point of the lamina (cardboard) quadrilateral $ABCD$, one would have anticipated that the four triangles subtended by G_L and the four sides would be equal in area. This not being the case in general as shown in Figure 1, therefore seems a bit counter-intuitive and unexpected.

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* The point mass centroid of a quadrilateral is located at the intersection of the lines connecting the midpoints of opposite sides.

† The lamina centroid of a quadrilateral is located at the intersection of the line connecting the centroids of triangles KLM and MNK with the line connecting the centroids of triangles KLN and LMN.

Area $\Delta AG_{pm}B$ = 13.58 cm^2
Area $\Delta BG_{pm}C$ = 8.24 cm^2
Area $\Delta CG_{pm}D$ = 16.37 cm^2
Area $\Delta DG_{pm}A$ = 21.70 cm^2

Area ΔKG_LL = 15.18 cm^2
Area ΔLG_LM = 18.78 cm^2
Area ΔMG_LN = 22.86 cm^2
Area ΔNG_LK = 21.29 cm^2

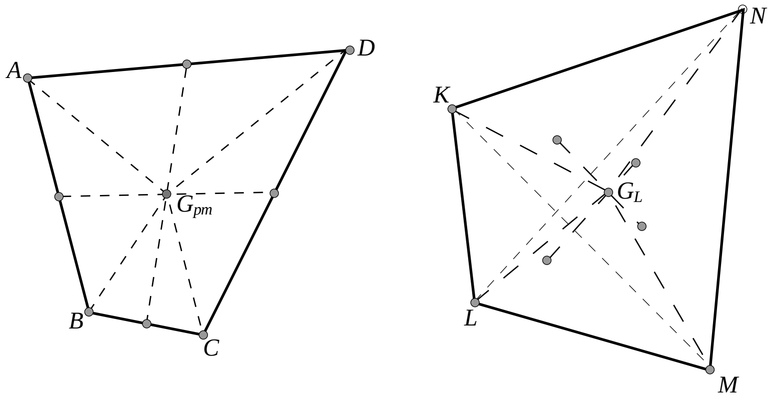

FIGURE 1*

The reader may now wish to use the following online dynamic sketch to explore experimentally where such a point *P* might be located for a general quadrilateral or some special cases:

http://dynamicmathematicslearning.com/equipartitioning-quad.html

Quite remarkably, such a (equipartitioning) point P that divides, or equipartitions, a quadrilateral into four triangles of equal area exists only for a bisect-diagonal quadrilateral. This follows from the following little known theorem proved by Pillay & Pillay [6] & Gilbert et al. [7, pp. 68-70]:

Theorem 2: A quadrilateral has an equipartitioning point *P* if, and only if, it is a bisect-diagonal quadrilateral, and then *P* is the midpoint of the diagonal bisecting the other.

The proof that the midpoint of the diagonal bisecting the other is the equipartitioning point *P* of a bisect-diagonal quadrilateral follows directly from Theorem 1, and is left to the reader. The following proof that only a bisect-diagonal quadrilateral has an equipartitioning point is slightly modified from that of [6] & [7], and is given below only for the convex case.

* On the previous page you have G_{PM} and here G_{pm}. Seems strange.

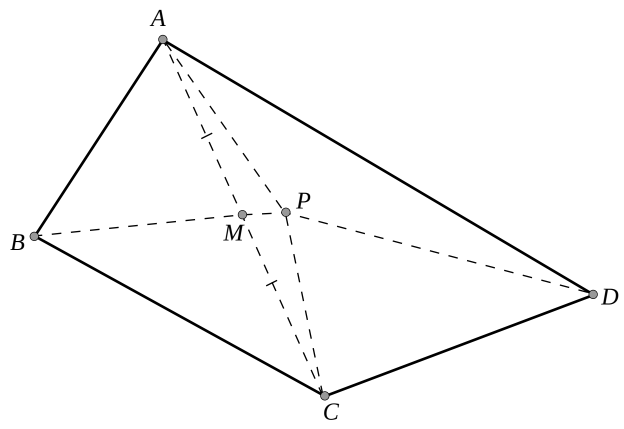

FIGURE 2

Proof: Suppose a convex quadrilateral *ABCD* has an equipartitioning point *P* as shown in Figure 2. Since it is given that triangles *APB*, *BPC*, *CPD* and *DPA* are equal in area, it follows that diagonals *BP* and *DP* bisect the areas of quadrilaterals *ABCP* and *APCD* respectively. Hence, from Theorem 1, both straight lines *BP* and *DP* extended contain the midpoint *M* of *AC*.

This implies that *DPM* is a straight line, and since the straight line through *M* and *P* must contain both *B* and *D* we conclude that *BMPD* must coincide with the diagonal *BD*, and that *BD* bisects *AC* in *M*. But triangles *APB* and *DPA* have the same area, so $BP = PD$. Thus we have shown that diagonal *BD* bisects diagonal *AC* and that the equipartitioning point *P* is the midpoint of *DB*.

Of course, the argument is entirely exchangeable, and we could in the same way argue that diagonal *AC* bisects diagonal *BD* and that the equipartitioning point *P* is the midpoint of *AC*. Either way, the result is proved that at least one of the diagonals of *ABCD* is bisected by the other.

The same argument, with a few modifications, applies when quadrilateral *ABCD* is concave, but is left to the reader. As shown in [7, pp. 69-70], one can also prove this theorem using a trigonometric argument that extends to the concave case.

Lamina and point mass centroids of a bisect-diagonal quadrilateral

Let us now examine the lamina and point mass centroids of a bisect-diagonal quadrilateral, and any relationship between them.

Given a bisect-diagonal quadrilateral *ABCD* as shown in Figure 4 with *M* the midpoint of the bisected diagonal *BD* and *P* the midpoint of diagonal *AC*. (According to Theorem 2, the point *P* is therefore the equipartitioning point of *ABCD*.)

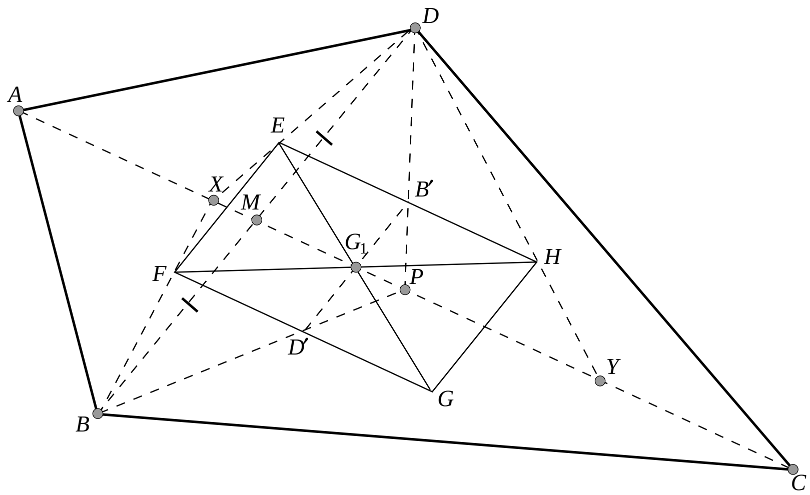

FIGURE 3

Theorem 3: Construct the centroids of triangles *DPA*, *APB*, *BPC* and *CPD* of a bisect-diagonal quadrilateral *ABCD* and label them respectively, *E*, *F*, *G* and *H*. Then *EFGH* is a parallelogram and the intersection of its diagonals, G_1, lies on *AC*, and is the lamina centroid of *ABCD*.

Proof: Since *E* lies on the median *DX* of triangle *DPA* and *H* lies on the median *DY* of triangle *CPD*, it follows that $EH \,//\, XY$ and $EH = \frac{2}{3}XY$. Similarly, $FG \,//\, XY$ and $FG = \frac{2}{3}XY$. Hence opposite sides *EH* and *FG* are parallel and equal, which shows that *EFGH* is a parallelogram. Since the areas of triangles *DPA*, *APB*, *BPC* and *CPD* are equal, the weight of their respective laminas are equally concentrated at their centroids; hence all together, their lamina weights balance at the intersection, G_1, of the diagonals of *EFGH*. Moreover, since *EH* and *FG* are the same distance away from *AC*, it follows that *AC* passes through the symmetrical point, G_1, of *EFGH*. This completes the proof of Theorem 3.

In addition, since $XP = PY$, note that B', the centroid of triangle *ACD*, is the midpoint of *EH*. Similarly, D' is the midpoint of *FG*. Since the centroids A' and C', respectively, of triangles *BCD* and *ABD*, lie on diagonal *AC*, the line $B'D'$ also intersects the line $A'C'$ (line *AC*) at the lamina centroid, G_1.

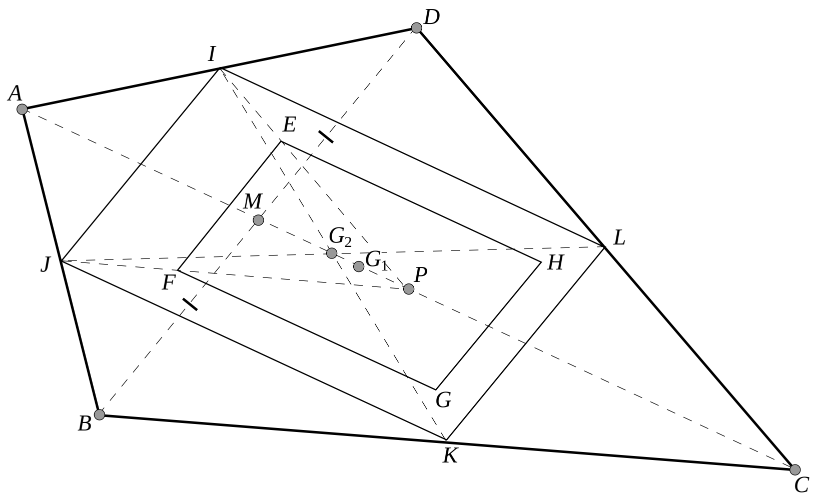

FIGURE 4

Theorem 4: The lamina parallelogram *EFGH* of a bisect-diagonal quadrilateral *ABCD* is homothetic to the Varignon parallelogram *IJKL* formed by the midpoints of the sides of *ABCD*, with the centre of similarity between the two located at *P*, and a scale factor of $\frac{2}{3}$.

Proof: Since *E* and *F* are the respective centroids of triangles *DPA* and *APB*, we have in triangle *IPJ* that $EF \,//\, IJ$ and $EF = \frac{2}{3}IJ$. Since the same can be shown for the other pairs of corresponding sides of *EFGH* and *IJKL*, it follows that *EFGH* is homothetic to *IJKL* with centre *P* and scale factor $\frac{2}{3}$.

Theorem 5: The distance between the lamina centroid G_1 and the equi-partitioning point *P* of a bisect-diagonal quadrilateral is twice that of the distance between its lamina centroid G_1 and point mass centroid G_2.

Proof: Since the point mass centroid G_2 is located at the intersection of the diagonals of the Varignon parallelogram *IJKL*, it follows from the similarity transformation in Theorem 4 that $G_1P = 2G_2G_1$.

In addition, according to a well-known result in [5, p. 54] and [1, p. 216] the point mass centroid G_2 also lies at the midpoint of the line segment *MP*. Hence $3G_2G_1 = G_2P \Rightarrow 6G_2G_1 = MP$.

The Newton-Gauss line

Since the celebrated Newton–Gauss line [8, p. 62] is the straight line containing the midpoints of the three diagonals of a complete quadrilateral, it immediately follows that the diagonal *AC* passes through the midpoint *S* of the third diagonal *QR* of the complete bisect-diagonal quadrilateral *ABCDQR*.

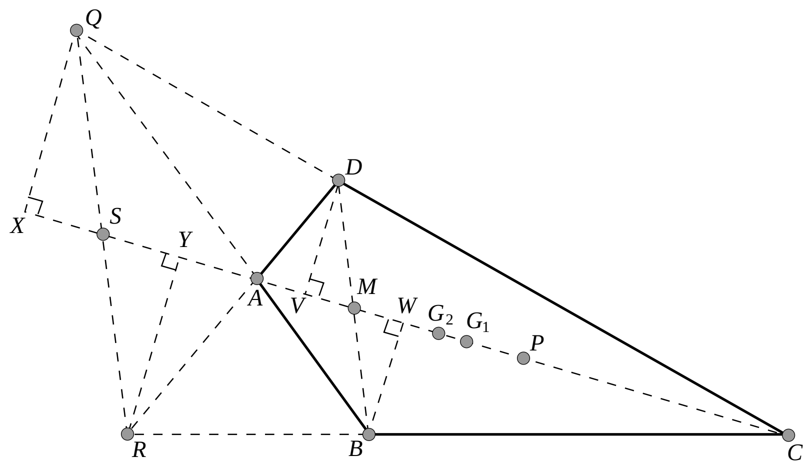

FIGURE 5

Theorem 6: Given a complete bisect-diagonal quadrilateral *ABCDQR* as shown in Figure 5 with diagonal *AC* bisecting diagonal *BD*, then the third diagonal *QR* is parallel to *BD*.

Proof: Drop perpendiculars from *Q*, *R*, *D* and *B* to *AC*. From the similarity of triangles *QXC* and *DVC* it follows that $\frac{CD}{CQ} = \frac{DV}{QX}$. Similarly, $\frac{CB}{CR} = \frac{BW}{RY}$. From the congruency of triangles *QXS* and *RYS*, and of triangles *DVM* and *BWN*, we have $\frac{DV}{QX} = \frac{BW}{RY}$. Hence $\frac{CD}{CQ} = \frac{CB}{CR}$, which implies that *QR* is parallel to *BD*.

Conversely, given a complete quadrilateral *ABCDQR* with diagonal *QR* parallel to *BD*, then it is easy to see that the above argument applies in reverse, and that diagonal *AC* will bisect diagonal *BD*. In other words, *ABCD* will be a bisect-diagonal quadrilateral.

Concluding comment

Apart from parallelograms and kites as special cases of a bisect-diagonal quadrilateral, it might also be of interest to some readers to note that that any cyclic quadrilateral *ABCD* with its sides *AB* : *BC* : *CD* : *DA* in geometric progression with common ratio *r*, as shown in [9], is also a bisect-diagonal quadrilateral. It is easy to establish and left as an exercise.

Note: A dynamic geometry sketch illustrating the properties of a bisect-diagonal quadrilateral explored here is available online at:

http://dynamicmathematicslearning.com/bisect-diagonal-quadrilateral.html

References

1. M. Josefsson, Properties of bisect-diagonal quadrilaterals, *Math. Gaz*. **101** (July 2017) pp. 214-226.
2. M. de Villiers, *Some adventures in Euclidean geometry*, Dynamic Mathematics Learning (2009).
3. G. Graumann, Investigating and ordering quadrilaterals and their analogies in space − problem fields with various aspects, *ZDM* **37** (3) (2005) pp. 190-198,

   also available at: http//subs.emis.de/journals/ZDM/zdm053a8.pdf
4. A. Ramachandran, The four-gon family tree, *At Right Angles* **1** (1) (2012) pp. 53-57, also available at http://www.teachersofindia.org/sites/default/files/12_four_gon_family_tree.pdf
5. H. S. M. Coxeter & S. L. Greitzer, *Geometry revisited*, Math. Ass. Amer. (1967).
6. S. Pillay & P. Pillay, Equipartitioning and balancing points of polygons, *Pythagoras*, **71** (July 2010) pp. 13-21.
7. G. T. Gilbert, M. Krusemeyer and L. C. Larson, The Wohascum County problem book, *Dolciani Mathematical Expositions*, **14**(10), Math. Ass. Amer. (1993).
8. R. A. Johnson, *Advanced Euclidean geometry*, Dover Publications (1960).
9. M. De Villiers, A cyclic Kepler quadrilateral & the Golden Ratio, At Right Angles, March 2018, pp. 91-94. (Accessed on 16 August 2020 at: https://azimpremjiuniversity.edu.in/SitePages/resources-ara-march-2018-cyclic-kepler-quadrilateral.aspx).

MICHAEL DE VILLIERS

Mathematics Education (RUMEUS),

University of Stellenbosch, South Africa

e-mail: *profmd@mweb.co.za*